

# Spectral dimension of Horava-Snyder spacetime and the $AdS_2 \times S^2$ momentum space

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We show that the UV-regime at the Lifshitz point  $z = 3$  is equivalent to work with a momenta manifold whose topology is the same as that of an  $AdS_2 \times S^2$  space. According to Snyder's theory, curved momentum space is related to non-commutative quantized spacetime. In this sense, our analysis suggests an equivalence between Horava-Lifshitz and Snyder's theory.

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## I. INTRODUCTION

There is the possibility of understanding the quantum gravity aspects by studying the spectral dimension of the spacetime as considered by Horava and Ambjorn [1–4]. One of the best way of applying such investigations is through the diffusion equation. The diffusion process can be seen as an way of a diffusing particle to probe the spectral dimension. It happens that the dimension seen by the particle can change along its diffusing process. It may even become fractal as in polymeric chains. In this letter we show that the spectral dimension of a curved momentum space gives the same result as in the Horava-Lifshitz gravity [1, 2]. In the latter case, for a 3+1-dimensional spacetime, i.e.,  $D = 3$ , the spectral dimension flows continuously from  $d_s = 2$  at  $z = 3$  to  $d_s = 4$  at  $z = 1$  as one goes from small to large distances. Although in the Horava-Lifshitz gravity the spacetime is continuous the behavior of the spectral dimension agrees with the Ambjorn's CDT quantum gravity [3, 4] that is based on a four-dimensional discrete spacetime — see also [5–7]. In our study we show that equivalently one can curve the momentum space to get the same spectral dimension in both Horava-Lifshitz gravity and CDT quantum gravity. This seems not to be surprising since it is well-known long ago [8, 9] that curved momentum space leads to discrete (and non-commutative) spacetime. Furthermore, as well discussed in [10] and anticipated by Snyder [8, 9], a quantized spacetime leads to *spacetime uncertainty relations* that follow from the algebra of coordinate operators describing the coordinates in quantum spacetime. Such uncertainties introduces limitations in the accuracy of localization of spacetime events in very short distances (or very high energies) — see below. This is the regime where a quantum theory of gravity should be implemented. In the Horava-Lifshitz gravity this regime is understood as a quantum theory at Lifshitz point  $z = 3$ . Another interesting candidate to quantum gravity is Doubly Special Relativity (DSR) which also develops spacetime noncommutativity that one can be shown through the use of a Snyder type algebra — see [11].

## II. THE DIFFUSION EQUATION AND THE CURVED SPACE MOMENTUM

The spectral dimension can be understood in terms of a diffusion equation. The diffusion time is regarded as the scale responsible to probe the manifold in study. At small diffusion time the dimension of a curved manifold coincides with the spectral dimension. At sufficiently large diffusion time they start to be different. In our investigations we assume the spacetime to be a flat manifold. This is because the spectral UV/IR flow in Horava-Lifshitz theory should still be true for curved spacetime as shown in [12].

One can consider the diffusion equation as

$$\frac{\partial}{\partial \sigma} \rho(\mathbf{x}, \tau; \mathbf{x}', \tau'; \sigma) = \left( \frac{\partial^2}{\partial \tau^2} + \nabla^2 \right) \rho(\mathbf{x}, \tau; \mathbf{x}', \tau'; \sigma) \quad (1)$$

whose solution is

$$\rho(\mathbf{x}, \tau; \mathbf{x}', \tau'; \sigma) = \int \frac{d\omega d^D \mathbf{k}}{(2\pi)^{D+1}} e^{i\omega(\tau-\tau') + i\mathbf{k} \cdot (\mathbf{x}-\mathbf{x}')} e^{-\sigma(\omega^2 + |\mathbf{k}|^2)} \quad (2)$$

This solution enables us to find the *average return probability*  $P(\sigma) \equiv \rho(\mathbf{x}, \tau; \mathbf{x}', \tau'; \sigma) \Big|_{\mathbf{x}=\mathbf{x}'; \tau=\tau'}$  given by

$$\begin{aligned} P(\sigma) &= \int \frac{d\omega d^D \mathbf{k}}{(2\pi)^{D+1}} e^{-\sigma(\omega^2 + |\mathbf{k}|^2)} \\ &= \frac{C}{\sigma^{(D+1)/2}}. \end{aligned} \quad (3)$$

The spectral dimension is given by

$$d_s = -2 \frac{d \ln P(\sigma)}{d \ln \sigma} = D + 1, \quad (4)$$

where  $C$  is some nonzero constant. In this case the spectral dimension coincides with the topological dimension of the  $\mathbb{R}^{D+1}$  spacetime [1].

In the Horava-Lifshitz gravity one modifies the UV behavior of theory by using the Lifshitz scaling such that  $|\mathbf{k}|^2 \rightarrow |\mathbf{k}|^{2z}$ , being  $z = 3$  the Lifshitz point at which gravity is renormalizable via power counting [1, 2]. The IR regime of the theory is recovered at  $z = 1$ . In this proposal one has

$$\begin{aligned} P(\sigma) &= \int \frac{d\omega d^D \mathbf{k}}{(2\pi)^{D+1}} e^{-\sigma(\omega^2 + |\mathbf{k}|^{2z})} \\ &= \frac{C}{\sigma^{(\frac{D}{z}+1)/2}}. \end{aligned} \quad (5)$$

The spectral dimension is now given by

$$d_s = \frac{D}{z} + 1, \quad (6)$$

that is the most important result found in [1]. Of course, in this setup one has to change the Laplacian  $\Delta$  of the diffusion equation (1) in the form  $\nabla^2 \rightarrow \nabla^{2z}$ . In a general curved manifold the formula (4) should be replaced by [12]

$$n = 2 \lim_{\lambda \rightarrow \infty} \frac{d \ln N_\Delta(\lambda)}{d \ln \lambda}, \quad (7)$$

where  $N_\Delta(\lambda)$  counts the number of eigenvalues, with multiplicity less than  $\lambda$ , of the Laplacian  $\Delta$  on a closed Riemannian manifold  $\mathcal{M}$  of dimension  $n$ . In this case the spectral dimension presents the same flow as developed in flat spacetime, that is  $n \equiv d_s$ . Thus,  $n = 2$  at UV regime flows to  $n = 4$  at IR regime — see [12] for further details. However, for the sake of simplicity, in the following we maintain our investigations in flat spacetime.

Notice we can rewrite the first equation in (5) in terms of a new  $D$ -dimensional momentum variable  $|\mathbf{k}| \rightarrow |\mathbf{p}|^{1/z}$  to find

$$P(\sigma) = \int \frac{d\omega d^D \mathbf{p}}{(2\pi)^{D+1}} f(|\mathbf{p}|, z) e^{-\sigma(\omega^2 + |\mathbf{p}|^2)}, \quad (8)$$

where we define the momentum dependent function as

$$f(|\mathbf{p}|, z) = c |\mathbf{p}|^\alpha, \quad \alpha = \frac{D}{z} - D. \quad (9)$$

In order to determine the spectral dimension on influence of the momentum dependent function, let us first obtain the average return probability in the form

$$\begin{aligned} P(\sigma) &= 4\pi c \int \frac{d\omega d^D \mathbf{p}}{(2\pi)^{D+1}} |\mathbf{p}|^{D-1+\alpha} e^{-\sigma(\omega^2 + |\mathbf{p}|^2)} \\ &= \frac{C}{\sigma^{(D+\alpha+1)/2}}. \end{aligned} \quad (10)$$

In this case, the spectral dimension of the spacetime is given by

$$d_s = D + \alpha + 1 = \frac{D}{z} + 1, \quad (11)$$

that is precisely the result (6) for the spectral dimension in the Horava-Lifshitz gravity. For a 3+1-dimensional spacetime we have  $D = 3$ , in this case the spectral dimension flows continuously from  $d_s = 2$  at  $z = 3$  to  $d_s = 4$  at  $z = 1$  as one goes from small to large distances.

Now we argue that one can also make a flow between the UV and IR regime by considering a *curved momentum space* and keeping the theory fixed in  $z = 3$ . Thus the anisotropic rescaling of the momentum element volume made in (8) is now given by

$$f(|\mathbf{p}|, z) \equiv c |\mathbf{p}|^\alpha \sqrt{|\det G|}, \quad (12)$$

where  $G_{\mu\nu}(\mathbf{p})$  is the metric in the momentum space [8, 9].

Assume for a theory in 3+1 dimensions in the UV regime ( $\mathbf{p} \rightarrow \infty$ ) we have  $f(|\mathbf{p}|, z) \sim |\mathbf{p}|^{-2}$ , that is  $\sqrt{|\det G|} \rightarrow \text{const.}$  This precisely happens to the volume element of an  $AdS_2 \times S^2$  momentum space given by the metric

$$ds^2 = -\frac{|\mathbf{p}|^2}{\Lambda^2} d\omega^2 + \frac{\Lambda^2}{|\mathbf{p}|^2} d\mathbf{p}^2 + \Lambda^2 d\Omega_2^2, \quad (13)$$

where  $|\mathbf{p}|^2 = p_1^2 + p_2^2 + p_3^2$  and  $\Lambda$  is the  $AdS_2$  and  $S^2$  momentum radius. Thus the theory in UV regime have an  $AdS_2 \times S^2$  curved momentum space. It is well-know long ago that the momentum space with constant curvature such as  $AdS$  (or  $dS$ ) space has a *non-commutative spacetime* counterpart [8, 9]. We shall turn to this point shortly.

To make the metric to flow continuously to the IR regime one can use a more general metric such as the four-dimensional Reissner-Nordström black hole metric on the momentum space

$$ds^2 = -\left(1 + \frac{|\mathbf{p}|}{\Lambda}\right)^2 d\omega^2 + \left(1 + \frac{|\mathbf{p}|}{\Lambda}\right)^{-2} (d\mathbf{p}^2 + |\mathbf{p}|^2 d\Omega_2^2), \quad (14)$$

that becomes flat in the IR regime ( $\mathbf{p} \rightarrow 0$ ) where one recovers  $f(\mathbf{p}, z) \rightarrow 1$ . Notice that we have taken the metric of an extremal Reissner-Nordström black hole solution of a four-dimensional spacetime to lead to a black hole solution into the four-dimensional space momentum earlier discussed by making use of the following suitable change

$$\left(1 + \frac{r_0}{r}\right)^{\pm 2} \rightarrow \left(1 + \frac{\mathbf{p}}{\Lambda}\right)^{\mp 2}. \quad (15)$$

Thus, in general we have a curved momentum volume element rescaled by the function

$$f(|\mathbf{p}|, z) \equiv c|\mathbf{p}|^{\alpha+2} \left(1 + \frac{|\mathbf{p}|}{\Lambda}\right)^{-2} = \left(1 + \frac{|\mathbf{p}|}{\Lambda}\right)^{-2}, \quad (16)$$

where we have considered  $z = 3$ ,  $D = 3$ , that means  $\alpha = -2$ , and  $c = 1$ . It is worth noticing that for  $\frac{|\mathbf{p}|}{\Lambda} \ll 1$  one can write  $f(|\mathbf{p}|, z) \equiv \exp\left(-\frac{2|\mathbf{p}|}{\Lambda}\right)$ .

Thus, as expected, for the function  $f(|\mathbf{p}|, z) = \left(1 + \frac{|\mathbf{p}|}{\Lambda}\right)^{-2}$  into the formula (8), the spectral dimension (4) flows from  $ds = 2$  to  $ds = 4$  as one goes from the UV ( $\sigma \rightarrow 0$ ) to IR ( $\sigma \rightarrow \infty$ ) regime as the Fig. 1 shows. This has the same behavior found in the Ref. [3].

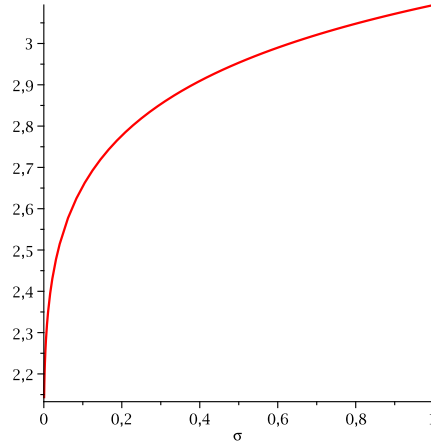


FIG. 1: The flow of the spectral dimension from  $ds = 2$  to  $ds = 4$  as one goes from UV ( $\sigma \rightarrow 0$ ) to IR ( $\sigma \rightarrow \infty$ ).

Let us now focus on the  $AdS$  part of the curved momentum space. Firstly, notice that we recover the  $AdS_2 \times S^2$  geometry (13) as the ‘near-horizon’ limit  $|\mathbf{p}| \rightarrow \infty$  of the metric (14) in momentum space. As firstly showed by Snyder the  $AdS$  momentum space is related to a non-commutative spacetime. In the following we shall make a short discussion on this important result in order to adapt it to our set up.

The  $AdS$  part of the momentum space with the  $AdS_2 \times S^2$  geometry satisfies

$$p_0^2 - |\mathbf{p}|^2 + p_4^2 = \Lambda^2, \quad p_0 = \frac{1}{a} \frac{\eta_0}{\eta_4}, \quad |\mathbf{p}| \equiv p_r = \frac{1}{a} \frac{\eta'}{\eta_4}, \quad p_4 = \frac{1}{a} \frac{\eta}{\eta_4}, \quad \Lambda^2 = \frac{1}{a^2}, \quad \eta' = |\vec{\eta}|, \quad (17)$$

where  $a$  (which is in the same footing as the Planck length  $\lambda_P$  given in [10]) is a natural unit of length of the quantized spacetime and the variables  $\eta_0, \dots, \eta_4$  satisfy the quadratic form that defines a four-dimensional space with constant curvature

$$-\eta^2 = \eta_0^2 - \eta'^2 - \eta_4^2, \quad (18)$$

where  $\eta^2 = \eta_1^2 + \eta_2^2 + \eta_3^2$ . This allows us to write commutation relations for a non-commutative spacetime whose coordinates are operators with the following structure

$$[\hat{t}, \hat{\mathbf{r}}] = i a^2 M_r, \quad M_r = \hat{\mathbf{r}} p_0 + \hat{t} p_r, \quad \hat{\mathbf{r}} = i a \left( \eta_4 \frac{\partial}{\partial \eta'} - \eta' \frac{\partial}{\partial \eta_4} \right), \quad \hat{t} = i a \left( \eta_4 \frac{\partial}{\partial \eta_0} + \eta_0 \frac{\partial}{\partial \eta_4} \right). \quad (19)$$

The commutation relation between the radial coordinate and its conjugate momentum is now given by

$$[\hat{\mathbf{r}}, p_r] = i(1 + a^2 p_r^2). \quad (20)$$

Notice that as  $a \rightarrow 0$  we recover the commutation relations of an ordinary commutative spacetime. On the other hand, at large momenta the non-commutativity of the spacetime and then the curvature of the momentum space become more evident. This is in accord with the earlier discussion on the four-dimensional Reissner-Nordström black hole metric on the momentum space in the ‘near-horizon’ limit  $|\mathbf{p}| \rightarrow \infty$  where this space becomes curved with  $AdS_2 \times S^2$  geometry.

### III. IR/UV (4D/2D) TRANSITION INTO LOOP MOMENTA

As in the previous discussions on the spectral dimension, we can also rewrite the following integral momenta in Horava-Lifshitz theory in terms of a new momentum variable  $|\mathbf{k}| \rightarrow |\mathbf{p}|^{1/z}$  such as

$$\int d\omega d^{D-1} \mathbf{k} \frac{1}{\omega^2 - |\mathbf{k}|^{2z} - M^2} \rightarrow \int d^D p \frac{f(|\mathbf{p}|, z)}{p^2 - M^2}, \quad (21)$$

for  $f(|\mathbf{p}|, z)$  given as in Eq. (9). Notice there is no difference in the physical description between the original and transformed integrals above. As a consequence we could change our way of facing the physics described by the integral of l.h.s. by looking into the integral of the r.h.s. recognizing it as the description of a *process* where a modified *loop momenta* via the function  $f(|\mathbf{p}|, z)$  is now present and the propagator is kept as the usual one. The mainly difference is that now the four-dimensional theory in the IR regime, i.e.,  $f(|\mathbf{p}|, z) = \left(1 + \frac{|\mathbf{p}|}{\Lambda}\right)^{-2} \rightarrow 1$  as  $|\mathbf{p}| \rightarrow 0$ , looks to be a two-dimensional theory in the UV regime, i.e.,  $f(|\mathbf{p}|, z) = \left(1 + \frac{|\mathbf{p}|}{\Lambda}\right)^{-2} \propto |\mathbf{p}|^{-2}$  as  $|\mathbf{p}| \rightarrow \infty$  — see below.

Here we show how to proceed in order to modify the usual loop momenta to get processes in the Horava-Lifshitz theory at UV-regime. Let us now apply the momentum dependent function  $f(|\mathbf{p}|, z)$  in a loop momenta integral that have quadratic divergence by power counting in a 3+1-dimensional spacetime in the following process given by the “tadpole”

$$\begin{aligned} i\lambda \int d^4 p \frac{1}{p^2 - M^2} &\rightarrow i\lambda \int d^4 p \frac{f(|\mathbf{p}|, z)}{p^2 - M^2} \\ &\rightarrow 4\pi c i\lambda \int d\omega \int_0^\infty d\mathbf{p} |\mathbf{p}|^{D-1+\alpha} \frac{1}{(\omega^2 - |\mathbf{p}|^2 - M^2)} \\ &\stackrel{UV}{\rightarrow} 4\pi c i\lambda \int d\omega \int_0^\infty d\mathbf{p} \frac{1}{(\omega^2 - |\mathbf{p}|^2 - M^2)} \end{aligned} \quad (22)$$

Notice that in the ultraviolet regime the integral (22) changes its quadratic divergence to logarithmic divergence. One should note that the integral into loop momenta is quite similar to an integral of a two-dimensional theory. This is in accord with the spectral flow observed above.

We shall not attempt to write down here a Lagrangian for a field theory with the UV-completion considered in Horava-Lifshitz theory, but our analysis suggests that it should be a renormalizable field theory that in UV regime behaves like a two-dimensional theory. This has a close connection with gravity in two dimensions. This is because two-dimensional gravity can be simply described in terms of a renormalizable two-dimensional field theory such as Liouville theory [13].

#### IV. DISCUSSIONS

In this letter we have found that in a curved momenta space with asymptotic  $AdS_2 \times S^2$  geometry one may have the same physics of Horava theory. Furthermore, if we allow ourselves to speculate a bit more, the would be holographic correspondence  $AdS_2/CFT_1$  in the momentum space would lead to a non-commutative conformal field theory in a one-dimensional spacetime, that may correspond to a non-commutative conformal ‘quantum mechanics’. However is not clear at all which symmetries are present in both momentum space and spacetime in the present case. A point in this direction is the fact that in crystallography the cubic lattice is identical to the reciprocal lattice, but further studies in this direction should be addressed elsewhere.

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- [1] P. Horava, Phys. Rev. Lett. **102**, 161301 (2009) [arXiv:0902.3657 [hep-th]].
  - [2] P. Horava, Phys. Rev. D **79**, 084008 (2009) [arXiv:0901.3775 [hep-th]].
  - [3] J. Ambjorn, J. Jurkiewicz and R. Loll, “Spectral dimension of the universe,” Phys. Rev. Lett. **95**, 171301 (2005) [arXiv:hep-th/0505113].
  - [4] J. Ambjorn, J. Jurkiewicz and R. Loll, Phys. Rev. Lett. **93**, 131301 (2004) [arXiv:hep-th/0404156].
  - [5] J. D. Correia and J. F. Wheeler, “The spectral dimension of non-generic branched polymer ensembles,” Phys. Lett. B **422**, 76 (1998) [arXiv:hep-th/9712058].
  - [6] J. F. Wheeler and J. Correia, Nucl. Phys. Proc. Suppl. **73**, 783 (1999) [arXiv:hep-lat/9808020].
  - [7] J. Ambjorn, J. Jurkiewicz and Y. Watabiki, Nucl. Phys. B **454**, 313 (1995) [arXiv:hep-lat/9507014].
  - [8] H. S. Snyder, Phys. Rev. **71**, 38 (1947).
  - [9] H. S. Snyder, Phys. Rev. **72**, 68 (1947).
  - [10] S. Doplicher, K. Fredenhagen and J. E. Roberts, Commun. Math. Phys. **172**, 187 (1995) [hep-th/0303037].
  - [11] J. Kowalski-Glikman and S. Nowak, Int. J. Mod. Phys. D **12**, 299 (2003) [hep-th/0204245].
  - [12] A. Pinzul, Class. Quant. Grav. **28**, 195005 (2011) [arXiv:1010.5831 [hep-th]].
  - [13] A. Zamolodchikov and A. Zamolodchikov, *Lectures on Liouville Theory and Matrix Models*.